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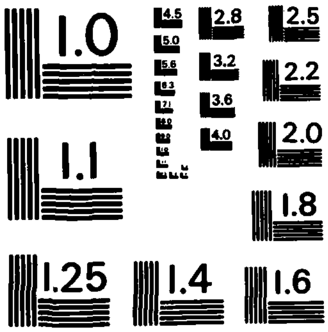

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ON STEADY VORTEX FLOW IN TWO DIMENSIONS, II

Bruce Turkington

Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53706

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MATHEMATICS RESEARCH CENTER

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ABSTRACT

We extend the results of our previous work (Part I) concerning steady ideal flow with vorticity in a bounded domain to the situation where the domain is unbounded and the flow is uniform at infinity. Prototypical examples include vortex pairs and Föppl vortex wakes behind a cylinder (in uniform translation). The existence of solutions and the asymptotic behavior of these solutions in a certain singular limit are established as in Part I using a direct variational method. The variational principle needed here is rather nonstandard, and so a detailed discussion of its formulation is given. Special difficulties arise for flows in an unbounded domain due to a lack of compactness (in the appropriate function space); consequently, we find that there is nonexistence of solutions in some cases.

AMS (MOS) Subject Classifications: 76C05, 35R35

Key Words: vorticity, variational inequality, asymptotic analysis, singular limit, point vortex, free boundary problem

Work Unit Number 1 (Applied Analysis)

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*Department of Mathematics, Northwestern University, Evanston, IL.

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SIGNIFICANCE AND EXPLANATION

The purpose of this sequel (to Part I) is to apply the results detailed in the previous work for a simple model case to a class of more complicated problems arising in the dynamics of a two-dimensional, ideal fluid. The theory of steady flow with vorticity developed in Part I is now extended to include certain flow geometries commonly occurring in practice - that is, steady ideal flows representing vortex pairs or (symmetric) vortical wakes behind a symmetric obstacle in a uniform stream. As before, special emphasis is given to flows with concentrated vorticity. A detailed discussion of the physical principles underlying the formulation of the rather novel variational approach to these problems is presented. Much of the analysis concerning the qualitative description of solutions parallels that of Part I, although some important differences must be dealt with here.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC and not with the author of this report.

ON STEADY VORTEX FLOW IN TWO DIMENSIONS, II

Bruce Turkington*

Department of Mathematics
Northwestern University
and
Mathematics Research Center
University of Wisconsin-Madison

In this sequel to Part I we continue the study of steady ideal fluid flows in two dimensions which possess vorticity. Specifically, we consider steady vortex flows imposed upon an underlying (non-trivial) irrotational flow. Classical examples of such flows include, for instance, the uniformly translating vortex pair and the Föppl vortex wake behind a cylinder in a uniform stream; it is the appropriate generalizations of flows such as these that we examine in the present paper. As in Part I we characterize solutions of the Euler fluid dynamical equations variationally as extremals for an appropriate energy functional of the vorticity. This approach permits us to develop here the natural extensions of both the existence and the

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asymptotic results given in Part I. In the present context, however, the variational problem itself is less familiar and therefore its formulation is discussed in detail. Particular emphasis is placed on unbounded fluid domains since special difficulties (not all of which are resolved here) arise in this case; for example, complementary existence and nonexistence theorems are proved for the vortex pair problem.

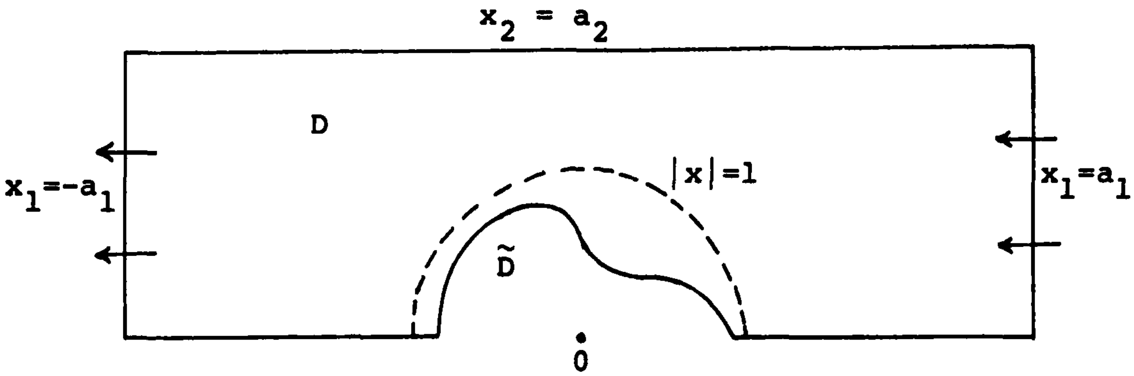
§1. VARIATIONAL PROBLEM

Let $\tilde{D} \subseteq R^2$ be the closure of a bounded, simply-connected subdomain of $R_+^2 = \{x=(x_1,x_2) \in R^2 : x_2 > 0\}$ for which $0 \in \partial\tilde{D}$. We shall assume by suitably scaling the x variables that $\tilde{D} \subseteq \{|x| < 1\}$. We shall study the flow of an ideal fluid with unit density in the (fluid) domain defined by

(1.1) $$D = \{x \in R_+^2 : |x_1| < a_1,\ x_2 < a_2\}\backslash \tilde{D}$$

for some (fixed) truncation parameters $1 < a_1, a_2 < \infty$. We shall assume that $\partial\tilde{D} \cap R_+^2$ is smooth, so that ∂D is piecewise smooth and has no re-entrant corners. We refer to $\tilde{D}$ as the obstacle. The flow geometry we have in mind is depicted in the figure.

The role of the truncation parameters $a = (a_1, a_2)$ is principally a technical one. It is of considerable interest (and in some ways more natural) to study flows in the unbounded domain $D = R_+^2\backslash\tilde{D}$;



however, certain aspects of the subsequent analysis are complicated in this case. Therefore, we choose to restrict our attention to truncated domains of the form (1.1) while developing the general results (§2-4). We then give more specialized results for flows in unbounded domains of the above type; in §5 flows past an obstacle $\tilde{D} \neq \phi$ are treated, while in §6 the vortex pair problem for which $D = R_+^2$ is studied. We remark that since $\partial D \cap \{x_2 = 0\}$ shall always be a streamline, all of these flows may be thought of as extended (by reflection) to the symmetric domain D^* formed by the union of D with its reflection about the x_1 axis.

The discussion in much of the sequel parallels that already given in Part I (namely [4]), and the notation is kept consistent with that of Part I. We shall proceed to formulate the variational problem to be studied in the present Part II, being brief at those points where the relevant details have been developed in Part I.

The (fluid) velocity field is given by

$$(1.2) \qquad u = (u_1(x), u_2(x)) = J\nabla\psi(x), \qquad x \in D$$

for some streamfunction ψ; recall $J(a_1,a_2) = (a_2,-a_1)$. The vorticity is then

$$(1.3) \qquad \omega = \omega(x) = -\Delta\psi(x), \qquad x \in D\ .$$

We seek flows satisfying the following boundary conditions:

$$(1.4) \qquad \begin{cases} \psi = -\mu & \text{on} \quad \partial\tilde{D} \\ \psi = -Qx_2 - \mu & \text{on} \quad \partial D \backslash \partial\tilde{D} \end{cases}$$

for constants $-\infty < \mu < \infty$ and $0 < Q < \infty$. This means in physical terms that we impose a flux across the boundary segments $\partial D \cap \{x_1 = \pm a_1\}$ with its rate prescribed by Q (the stream speed) while we require that the flow be tangential on the remainder of ∂D (as ψ is constant there). The value of ψ - defined only up to an additive constant - is normalized by the parameter μ. Clearly, these boundary conditions are chosen so that a flow in the truncated domain D approximates an unbounded flow past the obstacle $\tilde{D}$ with velocity $(-Q,0)$ at infinity.

The Green function for $-\Delta$ in D is denoted by

$$(1.5)\ g(x,x') = \frac{1}{2\pi} \log|x - x'|^{-1} - h(x,x'), \quad x,x' \in D\ ,$$

and the Green operator by

$$(1.6) \qquad G\omega(x) = \int_D g(x,x')\omega(x')dx'\ .$$

Also, we write

$$(1.7) \qquad H(x) = \frac{1}{2}\, h(x,x), \qquad x \in D \,.$$

Let $\eta = \eta(x)$, $x \in D$, denote the (unique) solution of

$$(1.8) \qquad \begin{cases} \Delta\eta = 0 & \text{in } D \\ \eta = 0 & \text{on } \partial\tilde{D} \\ \eta = x_2 & \text{on } \partial D \setminus \partial\tilde{D} \,; \end{cases}$$

then η is the streamfunction for a normalized irrotational flow in D. Now the streamfunction ψ can be expressed in terms of the triple ω, Q, μ, since according to (1.3) and (1.4) there must hold

$$(1.9) \qquad \psi = G\omega - Q\eta - \mu \,.$$

As explained in Part I, the steady Euler fluid dynamical equations can be expressed in weak form as

$$(1.10) \qquad \int_D \omega\,\partial(\psi,\phi)\,dx = 0 \quad \text{for all} \quad \phi \in C_0^\infty(D) \,;$$

recall $\partial(\psi,\phi) = \nabla\psi\cdot J\nabla\phi$. If Q is prescribed then this is a condition for ω alone. In fact, this is the condition that ω be a constrained extremal for a certain (energy) functional E_Q. Let the vortex energy be defined by the functional

$$(1.11) \qquad E(\omega) = \frac{1}{2}\int_D \omega(x)G\omega(x)\,dx$$

$$= \frac{1}{2}\int_D\int_D g(x,x')\omega(x)\omega(x')\,dx\,dx' \,,$$

and let the vortex impulse be defined by the functional

$$(1.12) \qquad P(\omega) = \int_D \eta(x)\omega(x)\,dx \,.$$

For any given $0 < Q < \infty$ we consider the functional

$$E_Q(\omega) = E(\omega) - QP(\omega) \ . \tag{1.13}$$

Then, as shown in Part I, the dynamical condition (1.10) is equivalent to

$$\frac{d}{dt} E_Q(\omega^{(t)})|_{t=0} = 0 \ ;$$

the variations $\omega^{(t)}$ of ω are defined by $\omega^{(t)}(x) = \omega(\xi_t^{-1}(x))$, $|t| < \tau$, where $x = \xi_t(y)$ is the measure preserving diffeomorphism of D defined by solving the equations

$$\frac{dx}{dt} = J\nabla\phi(x), \quad x(0) = y \ .$$

Thus we base our variational approach to the problem on the fact that any extremal ω of E_Q over a class of admissible functions containing all the variations $\omega^{(t)}$ yields a solution of (1.10) - that is, a dynamically possible flow.

The physical interpretations of E and P are discussed in detail in the appendix. Only in the special case of vortex pairs in which $D = R_+^2$ and $\eta(x) = x_2$ do these formulas seem to appear in the literature; for example, see Batchelor [1]. Of course, the corresponding concepts of energy and impulse are well developed in the standard literature for irrotational flows.

We let the class of admissible functions, $K_\lambda(D)$, be those $0 < \omega \in L^\infty(D)$ satisfying the constraints

$$(1.14) \qquad \int_D \omega(x)dx = 1$$

$$(1.15) \qquad \operatorname*{ess\,sup}_{x \in D} \omega(x) < \lambda .$$

The circulation in (1.14) is normalized to be 1 by scaling ω, and the vortex strength parameter, λ, is required to satisfy λ meas. $D > 1$. In §2 we give the basic existence result which states that there exists a maximizer $\omega = \omega_{Q,\lambda}$ for E_Q over the class $K_\lambda(D)$. Furthermore, the maximizer ω has the special form:

$$(1.16) \qquad \omega = \lambda I_\Omega, \ \Omega = \{x \in D : \psi(x) > 0\}$$

where ψ is defined by (1.9) and μ is a constant (uniquely determined by ω). There is no corresponding uniqueness result; indeed, nonuniqueness is present in some geometries.

In §3-4 we turn our attention to the limiting behavior of solutions as $\lambda \to \infty$. We show that

$$(1.17) \qquad \omega_{Q,\lambda}(x) \to \delta(x - X_Q^*) \quad \text{as} \quad \lambda \to \infty$$

in the sense of distributions; $\delta(z)$ is the unit (Dirac) delta measure at $z = 0$. The limit point X_Q^* is characterized by the property

$$(1.18) \qquad H_Q(X_Q^*) = \min_{x \in D} H_Q(x), \ H_Q(x) = H(x) + Q\eta(x) .$$

The precise nature of the limiting form of solutions is expressed in terms of their scaled versions:

$$\begin{aligned}(1.19)\qquad \zeta_{Q,\lambda}(y) &= \lambda^{-1}\omega_{Q,\lambda}(X_{Q,\lambda} + \varepsilon y)\\ &= I_{A_{Q,\lambda}} \qquad (A_{Q,\lambda} \subseteq R^2 \text{ is open and bounded})\end{aligned}$$

where $X_{Q,\lambda} = \int_D x\, \omega_{Q,\lambda}(x)dx$ and $\lambda\pi\varepsilon^2 = 1$. We prove that as $\lambda \to \infty$

$$(1.20)\quad \begin{cases} \zeta_{Q,\lambda} \to I_{B_1(0)} & \text{weakly star } L^\infty(R^2) \\ \partial A_{Q,\lambda} \to \partial B_1(0) & C^1 \text{ sense as curves .} \end{cases}$$

The methods used in §2-4 are exactly those already developed in the corresponding sections of Part I; consequently, these sections of the present Part II are brief.

New difficulties fundamentally different from those in Part I arise in §5-6 when unbounded domains are considered. Since then the functional E_Q is no longer weakly (star L^∞) continuous on $K_\lambda(D)$ we are forced to find solutions on truncated domains and to supply a priori estimates for their support. Only under the restriction that λ be large enough are we able to carry out such a procedure for the general case of a domain $D = R^2_+ \setminus \tilde{D}$; the previously established asymptotic properties of solutions are used crucially in the a priori estimates needed for the existence proof. This analysis constitutes §5. These steady vortex flows are the appropriate generalizations of the classical example due to Föppl of a point vortex pair stationary with respect to a moving cylinder (in the present notation $\tilde{D} = \{|x| < 1,\ x_2 > 0\}$ and $\lambda = \infty$); the Föppl vortices are referred to by Lamb [3] p. 223. The families of such flows (as constructed in §5) are of some real interest as they provide simple models for a familiar physical experience: the steady (symmetric) vortical wake behind a symmetric obstacle in a uniform stream.

The more tractable problem of (uniformly translating) vortex pairs in the special domain $D_0 = R_+^2$ is taken up in §6. In this case the vorticity possesses an additional symmetry: $\omega(-x_1,x_2) = \omega(x_1,x_2)$ and $\omega(x_1,x_2)$ is monotonic in x_1 for $x_1 > 0$. Various manifestations of this permit a more complete analysis of solutions than is possible in §5. A complete existence theory can be given but it requires that the class $K_\lambda(D_0)$ be slightly widened. In particular, the nonexistence of solutions (of the form (1.16) for a prescribed $Q > 0$) in $K_\lambda(D_0)$ is demonstrated for small enough λ; in terms of the flow this means that the vortex core cannot be too broadly spread. Finally, an alternate variational approach is sketched which closely resembles the approach used for vortex rings (the axisymmetric analogue) by Friedman and Turkington [2].

§2. EXISTENCE

Let D be as in (1.1) with a_1, a_2 fixed. Let $K_\lambda(D)$, the class of admissible functions in D, be defined by

$$(2.1) \qquad K_\lambda(D) = \{\omega \in L^\infty(D) : \int_D \omega(x)dx = 1, \\ 0 \le \omega(x) \le \lambda, \text{ a.e. } x \in D\} .$$

Let the (energy) functional E_Q be defined on $K_\lambda(D)$ by

$$(2.2) \qquad E_Q(\omega) = \frac{1}{2} \int_D \int_D g(x,x')\omega(x)\omega(x')dxdx' \\ - Q \int_D \eta(x)\omega(x)dx$$

for any given $0 < Q < \infty$; recall the notations established in (1.5-1.8). We assume throughout the sequel that

$$(2.3) \qquad \lambda > (\text{meas. } D)^{-1} .$$

The following existence theorem provides an absolute maximizer for E_Q in the class $K_\lambda(D)$.

<u>Theorem 2.1</u>. There exists $\omega = \omega_{Q,\lambda} \in K_\lambda(D)$ such that

$$(2.4) \qquad E_Q(\omega) = \max_{\tilde{\omega} \in K_\lambda(D)} E_Q(\tilde{\omega}) .$$

<u>Proof</u>. The proof is almost identical to the proof of Theorem 2.1 in Part I, except that E there is replaced by E_Q here. We leave the easy modifications to the reader.

The computations sketched in §1 (their more detailed counterparts are given in Part I) yield the following variational conditions.

Corollary 2.2. Whenever $\omega \in K_\lambda(D)$ satisfies (2.4) then there holds

$$(2.5)\qquad \int_D \omega\partial(G\omega - Q\eta,\phi)dx = 0 \quad \text{for all} \quad \phi \in C_0^\infty(D) .$$

The above functional dependence expressed in weak form can be given explicitly.

Corollary 2.3. Whenever $\omega \in K_\lambda(D)$ satisfies (2.4) then there exists a (uniquely determined) constant μ such that

$$(2.6)\qquad \omega = \lambda I_\Omega \quad \text{a.e. in} \quad D,$$

$$\Omega = \{x \in D : G\omega(x) - Q\eta(x) > \mu\} ;$$

I_Ω denotes the characteristic function of Ω.

Proof. Again, the proof is almost identical to the proof of Corollary 2.3 in Part I. Using the former methods we now find that

$$\mu = \operatorname*{ess\,sup}_{\omega(x)<\lambda}[G\omega(x) - Q\eta(x)] = \operatorname*{ess\,inf}_{\omega(x)>0}[G\omega(x) - Q\eta(x)] .$$

As before we then show that $\omega = 0$ a.e. in $\{G\omega - Q\eta = \mu\}$. We note that in contrast to the case in Part I μ is not necessarily positive.

§3. ASYMPTOTIC ESTIMATE

We let $\omega = \omega_{Q,\lambda}$ be any maximizer as in Theorem 2.1, and let ψ be given by (1.9).

Lemma 3.1. There holds

$$E_Q(\omega) \geq \frac{1}{4\pi} \log \frac{1}{\varepsilon} - C_1 \qquad (\lambda\pi\varepsilon^2 = 1) \tag{3.1}$$

where C_1 is a (positive) constant depending on Q.

Proof. We let $\hat{\omega} \in K_\lambda(D)$ be

$$\hat{\omega} = \lambda I_{B_\varepsilon(\hat{X}_Q)} \tag{3.2}$$

where $\hat{X}_Q$ is chosen such that

$$H_Q(\hat{X}_Q) = \min_{x \in D} H_Q(x), \quad H_Q(x) = H(x) + Q\eta(x) .$$

Then a straightforward calculation yields

$$E_Q(\omega) \geq E_Q(\hat{\omega}) \geq \frac{1}{4\pi} \log \frac{1}{2\varepsilon} - H_Q(\hat{X}_Q) + o(1)$$

as $\lambda \to \infty$, and so (3.1) follows.

The basic asymptotic estimate for large λ is proved by adapting the methods of Part I.

Theorem 3.2. There is a constant $R > 1$ depending on Q (independent of λ) such that

$$\operatorname{diam}(\operatorname{supp} \omega) \leq R\varepsilon \qquad (\lambda\pi\varepsilon^2 = 1) . \tag{3.3}$$

Proof. We define as in Part I

$$T(\omega) = \frac{1}{2} \int_D \psi\omega dx ; \tag{3.4}$$

then we have clearly an identity and an estimate:

(3.5) $$E(\omega) = T(\omega) + \frac{1}{2} QP(\omega) + \frac{1}{2} \mu ,$$

(3.6) $$\mu > 2E_Q(\omega) - 2T(\omega) .$$

We claim that $T(\omega) < C_2$ for a constant C_2 depending on Q (independent of λ). First we notice that $\mu > -Q/2(a_1 - 1)\lambda$; this follows since if $\mu < 0$ then $\{x : 1<|x_1|<a_1, Qx_2<|\mu|\} \subseteq \{x \in D : Q\eta(x) < |\mu|\}$ $\subseteq$ supp ω, and hence $2(a_1 - 1)|\mu|/Q < 1/\lambda$. Therefore, taking λ sufficiently large we may assume that $\mu > -\frac{1}{2}$. Now we estimate

$$2T(\omega) < \int_D (\psi - 1)^+\omega dx + 1 ;$$

but since $\psi < -\mu < \frac{1}{2}$ on ∂D we may apply the reasoning of Lemma 3.2 in Part I to conclude

$$\int_D |\nabla(\psi - 1)^+|^2 dx = \int_D (\psi - 1)^+\omega dx < C ,$$

and so the claimed bound for $T(\omega)$ is proved.

Returning now to (3.6) we have for arbitrary $x \in$ supp ω

$$G\omega(x) > G\omega(x) - Q\eta(x) > \mu > \frac{1}{2\pi} \log \frac{1}{\varepsilon} - C_3 ,$$

by virtue of Lemma 3.1. Hence we conclude

$$-2\pi C_3 < \int_D \log \frac{\varepsilon \operatorname{diam} D}{|x - x'|} \omega(x')dx' .$$

From this point on the proof is completed exactly as in the proof of Theorem 3.3 in Part I.

Remark. As is evident from the above proof the constant R depends on the diameter of D, and so the resulting estimate cannot be applied directly to solutions defined in unbounded domains.

§4. LIMITING BEHAVIOR

The precise limiting form of $\omega = \omega_{Q,\lambda}$ as $\lambda \to \infty$ is discussed in this section. By extracting a sequence $\lambda = \lambda_j \to \infty$ if necessary we suppose that

$$(4.1) \quad X_{Q,\lambda} = \int_D x\omega_{Q,\lambda}(x)dx \to X_Q^* \in \bar{D} \quad \text{as} \quad \lambda = \lambda_j \to \infty .$$

Theorem 4.1. Any X_Q^* as in (4.1) satisfies

$$(4.2) \qquad H_Q(X_Q^*) = \min_{x \in D} H_Q(x), \; H_Q(x) = H(x) + Q\eta(x) .$$

Proof. We follow the method of proof of Theorem 4.3 in Part I. Let $\hat{\omega} \in K_\lambda(D)$ be defined by (3.2). Then (recalling Lemma 4.2 in Part I) we have

$$E_Q(\hat{\omega}) \le E_Q(\omega) \le \frac{1}{4\pi} \int_D \int_D \log|x-x'|^{-1}\hat{\omega}(x)\hat{\omega}(x')dxdx'$$

$$- \frac{1}{2} \int_D \int_D h(x,x')\omega(x)\omega(x')dxdx' - Q \int_D \eta(x)\omega(x)dx.$$

Taking $\lambda \to \infty$ and using Theorem 3.2 we find, as required,

$$H_Q(X_Q^*) \le H_Q(\hat{X}_Q) = \min_{x \in D} H_Q(x) .$$

We are now able to assert that as $\lambda = \lambda_j \to \infty$

(4.3) $\omega_{Q,\lambda}(x) \to \delta(x - X_Q^*)$ in the sense of distributions

where $\delta(x - X_Q^*)$ is the unit (Dirac) delta measure at $x = X_Q^*$ given by (4.2). Moreover, the statements of Theorem 4.4 and Theorem 4.5 in Part I hold without any change for the scaled versions of $\omega_{Q,\lambda}$ and $\psi_{Q,\lambda}$ respectively; we will not restate them here. We note that the limiting functions ζ^* and v^* are, of course, independent of Q.

§5. UNBOUNDED DOMAINS

In this section we extend the theory previously developed in domains of the form (1.1) to unbounded domains

(5.1) $$D = R_+^2 \setminus \tilde{D} \qquad (\tilde{D} \neq \emptyset)$$

where $\tilde{D}$ is as before; that is, we study flows in the (full) exterior of a symmetric obstacle. We now write $D_0 = R_+^2$, $D_1 = R_+^2 \setminus \{|x| < 1\}$, and we assume (as before) that $D_1 \subseteq D \subseteq D_0$. Also, we denote by $g(x,x')$, $g_0(x,x')$, $g_1(x,x')$ the Green functions corresponding to D, D_0, D_1 respectively. The streamfunction η now represents irrotational flow past $\tilde{D}$ with velocity (1,0) at infinity, and is defined by

$$
(5.2)\quad \begin{cases} \Delta\eta = 0 \quad \text{in} \quad D \\ \eta = 0 \quad \text{on} \quad \partial D \\ \eta = x_2 + O(|x|^{-1}),\ \nabla\eta = (0,1) + O(|x|^{-2}) \\ \text{as } |x| \to \infty . \end{cases}
$$

The following lemma provides an a priori bound for the limiting location of a solution as $\lambda \to \infty$.

Lemma 5.1. Let $x^* \in D$ be an arbitrary critical point of $H_Q = H + Q\eta$, $0 < Q < \infty$. Then

$$
(5.3)\qquad |x_1^*| \le AQ^{-1}
$$

$$
(5.4)\qquad x_2^* \le AQ^{-1}
$$

for a constant A depending only on D.

Proof. The proof of these estimates requires some technical asymptotic properties of the functions H and η, so we present the demonstrations of these properties first.

Property of H: Let $\hat{H}$ be defined by

$$
(5.5)\qquad H(x) = \frac{1}{4\pi} \log \frac{1}{2x_2} + \hat{H}(x), \qquad x \in D .
$$

We claim that

$$
(5.6)\qquad |\nabla\hat{H}(x)| \le Cx_2|x|^{-4} \qquad (|x| > 2) .
$$

To prove this we observe that $\nabla\hat{H}(y) = \nabla_x\hat{h}(y,y)$, $y \in D$, if we define

$$
\hat{h}(x,x') = h(x,x') - \frac{1}{2\pi} \log|x - \overline{x'}|^{-1} ,
$$

where we write $\overline{x'} = (x_1', -x_2')$. For any fixed $x' \in D$, $\hat{h}(x,x')$ is harmonic in $x \in D$ and $\hat{h}(x,x') = 0$ when $x_2 = 0$. Thus, $\hat{h}(x,x')$ can be extended by reflection to be a harmonic function in $x \in D^* = R^2 \setminus (\tilde{D} \cup \tilde{D}^*)$ where $\tilde{D}^*$ is the reflection of $\tilde{D}$ about the x_1-axis; $\hat{h}(x,x')$ is then an odd function of x_2. It follows that the gradient is estimated by

$$(5.7)\quad |\nabla_x \hat{h}(y,y)| < C|y|^{-1} \sup_{|x-y|<\frac{1}{2}|y|} |\hat{h}(x,y)| \quad (|y| > 2).$$

In order to bound the right hand side of this inequality we introduce $\hat{h}_1(x,x')$ defined to be the corresponding function for the domain $D_1 = R_2^+ \setminus \{|x| < 1\}$; it is computed explicitly to be

$$\hat{h}_1(x,x') = \frac{1}{4\pi} \log(1 + 4x_2x_2'|x'|^{-2}|x - x'^*|^{-2}) ,$$

where we write $x'^* = |x'|^{-2}x'$. Now since $g_1(x,x') < g(x,x') < g_0(x,x')$ for $x,x' \in D_1$, we find that $0 < \hat{h}(x,x') < \hat{h}_1(x,x')$ for $x,x' \in D_1$. Therefore, the desired estimate (5.6) follows from (5.7) as we verify that

$$\sup_{|x-y|<\frac{1}{2}|y|} \hat{h}_1(x,y) < Cy_2|y|^{-3} .$$

Properties of η: Let $\hat{\eta}$ be defined by

$$(5.8)\qquad \eta(x) = x_2 + \hat{\eta}(x), \qquad x \in D .$$

We intend to determine expansions for $\hat{\eta}$ and $\nabla\hat{\eta}$ for large x; we proceed as usual from the Green representation formula

$$(5.9)\quad \hat{\eta}(x) = \int_{\partial\tilde{D}} [g_0(x,x')\,\frac{\partial\hat{\eta}}{\partial\nu'}(x') - \hat{\eta}(x')\,\frac{\partial g_0}{\partial\nu'}(x,x')]ds',$$

valid according to (5.2). Expanding $g_0(x,x')$, $|x| > 1$, in a Taylor series (up to second order) in x' about $x' = 0$ we obtain

$$(5.10)\qquad \hat{\eta}(x) = 2C_2x_2|x|^{-2} + 4C_{12}x_1x_2|x|^{-4} + O(|x|^{-3})$$

with

$$C_2 = \frac{1}{2\pi}\int_{\partial\tilde{D}} [x_2'\,\frac{\partial\hat{\eta}}{\partial\nu'}\,(x') - \hat{\eta}(x')\,\frac{\partial x_2'}{\partial\nu'}]ds'$$

and C_{12} given similarly ($x_1'x_2'$ replacing x_2' in the integral); the other corresponding coefficients C_1, C_{11}, C_{22} are all zero. Analogously, first differentiating (5.9) and then expanding, we get

$$(5.11)\quad \nabla\hat{\eta}(x) = \nabla(2C_2x_2|x|^{-2} + 4C_{12}x_1x_2|x|^{-4}) + O(|x|^{-4})\ .$$

We claim that $C_2 < 0$ (provided $\tilde{D} \neq \emptyset$). Indeed, using the fact that $\hat{\eta} = -x_2$ on $\partial\tilde{D}$, we find

$$\begin{aligned}(5.12)\qquad -2\pi C_2 &= \int_{\partial\tilde{D}} [-x_2\,\frac{\partial\hat{\eta}}{\partial\nu} + \hat{\eta}\,\frac{\partial x_2}{\partial\nu}]ds \\ &= \int_{\partial\tilde{D}} [\hat{\eta}\,\frac{\partial\hat{\eta}}{\partial\nu} - x_2\,\frac{\partial x_2}{\partial\nu}]ds \\ &= \int_D |\nabla\hat{\eta}|^2dx + \int_{\tilde{D}} |\nabla x_2|^2dx \\ &= M + \text{meas. }\tilde{D}\end{aligned}$$

where $M = \int_D |\nabla(\eta - x_2)|^2$ is the (so-called) induced mass of $\tilde{D}$.

The particular consequence of these derivations needed in the subsequent proof is the expansion:

(5.13) $\frac{1}{x_2} \eta_{x_1}(x) = -4C_2 x_1 |x|^{-4} + O(|x|^{-4})$ as $|x_1| \to \infty$.

<u>Proof of (5.4)</u>. By the above we have as $x_2 \to \infty$

$$\nabla H(x) = \left(0, \frac{-1}{4\pi x_2}\right) + \nabla \hat{H}(x), \quad |\nabla \hat{H}(x)| = O(|x|^{-3}) ,$$

$$\nabla \eta(x) = (0,1) + \nabla \hat{\eta}(x), \quad |\nabla \hat{\eta}(x)| = O(|x|^{-2}) .$$

Together these give

(5.14) $\nabla H(x) \cdot \nabla \eta(x) = \frac{-1}{4\pi x_2} + O(|x|^{-3})$ as $x_2 \to \infty$.

But at a critical point $x = x^*$ there holds

$$\nabla H(x^*) \cdot \nabla \eta(x^*) = -Q|\nabla \eta(x^*)|^2 = -Q + O(|x|^{-4}) ;$$

clearly, this contradicts (5.14) if $x_2^* Q > A$ for a sufficiently large constant A.

<u>Proof of (5.3)</u>. For this part we calculate

$$\nabla H(x) \cdot J\nabla \eta(x) = \frac{1}{4\pi x_2} \eta_{x_1}(x) + \nabla \hat{H}(x) \cdot J\nabla \eta(x) .$$

As $|x_1| \to \infty$ for $x_2 < AQ^{-1}$ we get, using (5.13), (5.6) and (5.2),

(5.15) $\nabla H(x) \cdot J\nabla \eta(x) = -\frac{1}{\pi} C_2 x_1 |x|^{-4} + O(Q^{-1}|x|^{-4})$.

But at a critical point $x = x^*$ there holds

$$\nabla H(x^*) \cdot J\nabla \eta(x^*) = -Q\nabla \eta(x^*) \cdot J\nabla \eta(x^*) = 0 ;$$

this contradicts (5.15) if $|x_1^*|Q > A$ for a sufficiently large constant A, since $C_2 \neq 0$.

In the next theorem we find that maximizers over a truncated class yield solutions in the (entire) domain D provided that λ is large. Let
$D^a = D \cap \{|x_1| < a_1,\ x_2 < a_2\}$ and let the class $K_\lambda^a(D)$ consist of all those $\omega \in K_\lambda(D)$ such that $\omega = 0$ a.e. in $D \backslash D^a$.

<u>Theorem 5.2</u>. Let $a = (a_1, a_2)$ be fixed so that $a_1, a_2 > 2\max\{1, AQ^{-1}\}$ with A as in Lemma 5.1. Then any (absolute) maximizer $\omega = \omega_{Q,\lambda}^a$ for E_Q over $K_\lambda^a(D)$ satisfies

(5.16) $$\operatorname{supp}\omega \subseteq \{|x_1| < a_1 - 1,\ 0 < x_2 < a_2 - 1\}$$
$$\text{provided } \lambda > \Lambda$$

where Λ is a sufficiently large constant depending on a, Q, D.

<u>Proof</u>. The existence of a maximizer ω follows by the proof of Theorem 2.1. The basic asymptotic estimate (3.3) with a constant R depending on a, Q, D holds for any such maximizer ω; the different method of truncation used here in §5 does not affect the validity of the proof given in §3. Taking Λ large enough so that $\lambda > \Lambda$ implies $R\varepsilon_\lambda < \frac{1}{2}$ and $|X_{Q,\lambda} - X_Q^*| < \frac{1}{2}$

(recalling (4.1) which is valid in the present case also) we find that

$$\operatorname{supp} \omega \subseteq B_1(X_Q^*)$$

for some minimum point X_Q^* of H_Q. Now, the required containment (5.16) follows immediately from Lemma 5.1.

The purpose of (5.16) is, of course, to ensure that the free boundary belonging to a maximizer ω does not touch the truncating boundary where $x_1 = \pm a_1$ or $x_2 = a_2$. It is possible then to proceed as in §3 to the conclusion that for ω given by Theorem 5.2 there holds

$$0 = \int_D \omega \partial(G\omega - Q\eta, \phi)dx \quad \text{for all} \quad \phi \in C_0^\infty(D) ,$$

and (for a uniquely determined constant μ)

$$\omega = \lambda I_\Omega \quad \text{a.e. in} \quad D, \ \Omega = \{x \in D : G\omega(x) - Q\eta(x) > \mu\}.$$

In short, $\omega = \omega_{Q,\lambda}^a$ (with $\lambda > \Lambda$) yields a solution (i.e. a dynamically possible steady flow) in the entire domain D even though the class of admissible functions is restricted to those supported in the truncated domain D^a.

We do not have an existence theory in D without the restriction that λ be large because the necessary a priori estimates on the support of truncated solutions are available (to us) only for large λ.

All of the statements given in §4 concerning the asymptotic behavior of solutions as $\lambda \to \infty$ carry over to the solutions found in Theorem 5.2. In particular, the location X_Q^* of the limiting point vortex is determined by (4.2) and hence satisfies the conclusions of Lemma 5.1. That is, it represents an equilibrium point for the Hamiltonian system

$$\frac{dX}{dt} = -J\nabla H_Q(X) ,$$

where $X(t)$ describes the motion of a (unit) point vortex moving in the fluid domain D in the presence of an irrotational flow with streamfunction $-Q\eta$. In the special case when $D = D_1$ (i.e. flow past a cylinder) the equilibrium point $x = X_Q^*$ has been computed explicitly by Föppl using the method of images; it is found to satisfy the equations

$$2|x|x_2 = |x|^2 - 1, \quad 1 = 2Qx_2(1 - |x|^{-4}) .$$

§6. VORTEX PAIRS

The most well-known example of steady vortex flow in R^2 is that due to a uniformly translating vortex pair. Taking account of the odd symmetry $(\omega(x_1,-x_2) = \omega(x_1,x_2))$ of ω on R^2 this problem corresponds to the special case when $D = D_0 = \{x \in R^2 : x_2 > 0\}$ in the present context. Furthermore, solutions also always possess the natural symmetry condition

(6.1) $$\omega(-x_1,x_2) = \omega(x_1,x_2) ;$$

throughout this section we append this additional requirement to the definition of the class of admissible functions $K_\lambda(D_0)$. The (energy) functional E_Q is defined on $K_\lambda(D_0)$ by

$$(6.2) \qquad E_Q(\omega) = \frac{1}{2}\int_{D_0}\int_{D_0} g_0(x,x')\omega(x)\omega(x')dxdx' - Q\int_{D_0} x_2\omega(x)dx$$

where the Green function is

$$(6.3) \quad g_0(x,x') = \frac{1}{4\pi}\log\frac{(x_1 - x_1')^2 + (x_2 + x_2')^2}{(x_1 - x_1')^2 + (x_2 - x_2')^2}\,.$$

We shall exploit the especially simple form of the problem on D_0 in order to give sharper results than can be given for the general problems of §5.

For any $\omega \in K_\lambda(D_0)$ we define its symmetrization in the x_1 variable, $\omega^* \in K_\lambda(D_0)$, in the usual way; namely,

$$\omega^*(x_1,x_2) = \omega^*(-x_1,x_2) = \Phi_\omega^{-1}(x_1;x_2) \qquad (x_1 > 0)$$

where $\Phi_\omega(t;x_2)$ (a monotonic function of t) is

$$\Phi_\omega(t;x_2) = \text{meas.}\{x_1' \in [0,\infty) : \omega(x_1',x_2) < t\}\,.$$

We observe the important fact that $E_Q(\omega^*) > E_Q(\omega)$; this follows since $g_0(x,x')$ is a decreasing function of $(x_1 - x_1')$ for fixed x_2,x_2' and hence

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g_0(x,x')\omega^*(x)\omega^*(x')dx_1dx_1'$$

$$> \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g_0(x,x')\omega(x)\omega(x')dx_1dx_1'\,,$$

a consequence of standard theory. Therefore, any maximizer ω for E_Q over $K_\lambda(D_0)$ can be assumed to be symmetrized, $\omega = \omega^*$.

We begin our analysis of vortex pairs by stating an existence theorem analogous to that given in §5, providing solutions in D_0 for large λ.

<u>Theorem 6.1</u>. Let $a = (a_1, a_2)$ be fixed so that $a_1, a_2 > 2 \max\{1, 1/4\pi Q\}$. Then any (absolute) maximizer $\omega = \omega^a_{Q,\lambda}$ for E_Q over $K^a_\lambda(D)$ satisfies

$$(6.4) \qquad \operatorname{supp} \omega \subseteq \{|x_1| < a_1 - 1,\ 0 < x_2 < a_2 - 1\}$$

provided $\lambda > \Lambda$

where Λ is a sufficiently large constant depending on a, Q.

<u>Proof</u>. We need only modify the proof of Theorem 5.2. By virtue of condition (6.1) and the fact that

$$H_0(x) = \frac{1}{4\pi} \log \frac{1}{2x_2},$$

the only critical point $x = X^*_Q$ of $H_0(x) + Qx_2$ is $X^*_Q = (0, 1/4\pi Q)$. As before we find that $\operatorname{supp} \omega \subseteq B_1(X^*_Q)$ if $\lambda > \Lambda$ for a sufficiently large constant Λ, and this yields (6.4).

The above theorem ensures that the maximizers $\omega = \omega^a_{Q,\lambda}$ with $\lambda > \Lambda$ represent solutions of the vortex pair problem in D_0. These solutions, being symmetrized, then have the form

$$(6.5) \qquad \omega = \lambda I_\Omega, \ \Omega = \{x \in D_0 : G_0\omega(x) - Qx_2 > \mu\}$$

with $\Omega = \{|x_1| < z(x_2)\}$ for a function z defined implicitly by $\psi(z(x_2),x_2) = 0$. It is straightforward (using $\omega = \omega^*$) to verify that $\psi_{x_2}(x) < 0$ for $x_1 > 0$, and thus $z \in C^1$ on those open intervals where $z > 0$.

The requirement in Theorem 6.1 that λ be large is more restrictive than the variational theory should require and is imposed mainly for technical reasons. On the other hand, the following theorem shows that solutions in $K_\lambda(D_0)$ for arbitrarily small λ are not possible.

<u>Theorem 6.2</u>. There does not exist $\omega = \omega^* \in K_\lambda(D_0)$ such that (6.5) holds (for some constant μ) if λ is sufficiently small depending on Q, $0 < Q < \infty$.

In the proof of this nonexistence result we require three lemmas concerning a priori estimation of the support of solutions having the form (6.5); these lemmas are presented first. We remark that in (6.5) it is necessary that $\mu > 0$; otherwise, the containment $\{0 < x_2 < |\mu|/Q\} \subseteq \Omega$ contradicts the constraint meas. $\Omega = 1/\lambda < \infty$.

Lemma 6.3. For all $x \in \operatorname{supp} \omega$ there holds

$$(6.6) \qquad Q^2|x_1|x_2 < E(\omega) = \frac{1}{2}\int_{D_0} \omega(x)G_0\omega(x)dx \ .$$

Proof. Let $x \in \operatorname{supp} \omega$; we observe that $(x_1', x_2) \in \operatorname{supp} \omega$ whenever $|x_1'| < |x_1|$ by virtue of the symmetrization $\omega = \omega^*$. Since $\mu > 0$ we then have

$$Qx_2 < G_0\omega(x_1', x_2) < \int_0^{x_2} |\nabla G_0\omega(x_1', x_2')|dx_2' \ .$$

Upon integration in x_1' this yields

$$\begin{aligned} 2Q|x_1|x_2 &< \int_{\substack{|x_1'|<|x_1| \\ 0<x_2'<x_2}} |\nabla G_0\omega(x')|dx' \\ &< (2|x_1|x_2)^{1/2}\left(\int_{D_0} |\nabla G_0\omega(x')|^2 dx'\right)^{1/2} \\ &= 2(|x_1|x_2E(\omega))^{1/2} , \end{aligned}$$

as required.

In the results to follow we make use of the elementary estimates

$$(6.7) \qquad g_0(x,x') < \frac{1}{2\pi}\log\left(1 + \frac{2x_2}{|x-x'|}\right)$$

$$< \begin{cases} C_0 \log \dfrac{x_2}{|x-x'|} & \text{if } |x-x'| < x_2/2 \\[2ex] C_0 \dfrac{x_2}{|x-x'|} & \text{if } |x-x'| > x_2/2 \end{cases}$$

for a certain absolute constant C_0.

Lemma 6.4. For all $x \in \text{supp}\,\omega$ there holds

$$x_2 \leq CQ^{-1} \max\{1, \log \lambda^{1/2} x_2\} . \tag{6.8}$$

Proof. Let $\lambda\pi\rho_\lambda^2 = 1$ define $\rho_\lambda > 0$. If $\rho_\lambda \leq x_2/2$, then

$$G_0\omega(x) \leq C_0 \int_{|x-x'|<\rho_\lambda} \log \frac{x_2}{|x-x'|} \lambda dx' \leq C \log(x_2/\rho_\lambda).$$

If $\rho_\lambda > x_2/2$, then

$$G_0\omega(x) \leq C_0 \int_{|x-x'|<x_2/2} \log \frac{x_2}{|x-x'|} \lambda dx'$$

$$+ C_0 x_2 \int_{x_2/2<|x-x'|<\rho_\lambda} |x-x'|^{-1} \lambda dx'$$

$$\leq C\lambda x_2 \rho_\lambda .$$

In either case we conclude

$$Qx_2 \leq G_0\omega(x) \leq C \max\{1, \log(x_2/\rho_\lambda)\} ,$$

and so (6.8) is proved.

Lemma 6.5. For all $x \in \text{supp}\,\omega$ there holds

$$|x_1| \leq CQ^{-1}[1 + \frac{\lambda E(\omega)}{Q^2} \max\{1, \log \frac{Q^2|x_1|}{\lambda^{1/2} E(\omega)}\}] . \tag{6.9}$$

Proof. Recalling (6.6) we define $y_2 = 2E(\omega)/Q^2|x_1| > 2x_2$; then whenever $x' \in \text{supp}\,\omega$ with $x_2' > y_2$ there must hold $|x_1'| < |x_1|/2$. In light of this we have

$$(6.10) \qquad Qx_2 < G_0\omega(x) < \int_{0<x_2'<y_2} g_0(x,x')\omega(x')dx' + \int_{|x_1'|<|x_1|/2} g_0(x,x')\omega(x')dx' .$$

We proceed to estimate these integrals separately.

First term:

$$\int_{0<x_2'<y_2} g_0(x,x')\omega(x')dx'$$

$$< C_0 \int_{|x-x'|<x_2/2} \log \frac{x_2}{|x-x'|} \lambda dx'$$

$$+ C_0 x_2 \int_{\substack{|x-x'|>x_2/2 \\ 0<x_2'<y_2}} |x-x'|^{-1}\omega(x')dx'$$

$$< C\lambda x_2^2 + C\lambda x_2 y_2 \max\{1, \log(1/\lambda^{1/2} y_2)\}$$
$$< C\lambda x_2 y_2 \max\{1, \log(1/\lambda^{1/2} y_2)\} ;$$

here (in the second member) we observe that for $|x - x'| > x_2/2$ there holds $|x - x'|^{-1} < 5|\bar{x} - x'|^{-1}$, and thus, using $\omega \in K_\lambda(D_0)$, we need only estimate

$$\int_{\substack{|x_1-x_1'|<2\max\{y_2, 1/\lambda y_2\} \\ 0<x_2'<y_2}} |\bar{x}-x'|^{-1}\lambda dx'$$

$$< Cy_2 \max\{1, \log(1/\lambda^{1/2} y_2)\} .$$

Second term:

$$\int_{|x_1'|<|x_1|/2} g_0(x,x')\omega(x')dx'$$

$$< C_0x_2 \int_{|x_1'|<|x_1|/2} |x-x'|^{-1}\omega(x')dx'$$

$$< Cx_2|x_1|^{-1} .$$

Returning with these estimates to (6.10) we obtain

$$Qx_2 < Cx_2[\lambda y_2\max\{1,\log(1/\lambda^{1/2} y_2)\} + |x_1|^{-1}] ;$$

the desired result (6.9) now follows by the definition of y_2.

Proof of Theorem 6.2. We assume that $0 < \lambda < 1$. Lemma 6.4 implies that there is a bound $b_2 = b_2(Q)$ such that $x_2 < b_2$ for $x \in \operatorname{supp} \omega$. With $b_2 > 1$ fixed we may estimate the energy

$$0 < E(\omega) < \sup_{\Omega} G_0\omega < C(Q)\lambda \log 2/\lambda ;$$

to see this we apply the reasoning used in the proof of Lemma 6.5 (with b_2 replacing y_2) to get

$$G_0\omega(x) = \int_{0<x_2'<b_2} g_0(x,x')\omega(x')dx'$$

$$< C\lambda x_2b_2\max\{1,\log(1/\lambda^{1/2} b_2)\}$$
$$< C\lambda b_2^2\log(2/\lambda^{1/2} b_2) .$$

Lemma 6.5 now implies that there is a bound $b_1 = b_1(Q)$ such that $|x_1| < b_1$ for $x \in \operatorname{supp} \omega$. Obviously, these bounds (independent of λ) for the support of ω together give

$$\int_{D_0} \omega(x)dx < 2\lambda b_1 b_2 < 1$$

provided λ is sufficiently small (depending on Q), and this is a contradiction since $\omega \in K_\lambda(D_0)$.

In view of the above nonexistence result it is desirable to widen the class of admissible functions in order to be able to give a variational theory without the restriction that λ be large. We therefore consider the new class

$$K'_\lambda(D_0) = \{\omega : \int_{D_0} \omega dx \le 1,\ 0 \le \omega \le \lambda \text{ a.e. in } D_0\} ,$$

where we also continue to impose (6.1). The existence of an absolute maximizer $\omega = \omega^*$ for E_Q over $K'_\lambda(D_0)$ can be proved for arbitrary $0 < \lambda < \infty$ using the methods of this paper. The support of any such ω can be bounded (depending on Q) using the proofs of Theorem 6.1 and Theorem 6.2 respectively for the cases of large λ and small λ. Also, as we expect,

$$\int_{D_0} \omega dx \begin{cases} = 1 & \lambda \text{ large} \\ < 1 & \lambda \text{ small} , \end{cases}$$

although a sharp estimate for the value of λ separating these two cases is not clear. We omit the details here.

We close our discussion of vortex pairs with a description of an alternate variational approach to the problem. Rather than prescribing the speed Q (in the functional E_Q) we prescribe instead the vortex impulse $P(\omega)$; then Q must be determined along with the extremal ω as the Lagrange mul+iplier corresponding to the vortex impulse constraint. The variational problem is then to maximize the energy functional

$$E(\omega) = \frac{1}{2} \int_{D_0} \omega(x) G_0 \omega(x) dx$$

subject to the constraint

$$\int_{D_0} x_2 \omega(x) dx = P \qquad (0 < P < \infty)$$

over the class of admissible functions $\omega \in K'_\lambda(D_0)$.

The existence of a maximizer $\omega = \omega^* = \omega_{P,\lambda}$ can be proven for arbitrary $0 < \lambda < \infty$, $0 < P < \infty$, and any such solution can be shown to have the form (6.5) for some $Q > 0$, $\mu > 0$. This alternate approach has some particular advantages when used in the analogous (axisymmetric) problem of vortex rings, and has been worked out in detail in Friedman and Turkington [2]; the variational formulation itself is an adaptation of that proposed by Benjamin.

The speed Q is determined by ω as follows:

$$(6.11) \qquad Q \int_D \omega(x) dx = \frac{1}{2\pi} \int_{D_0} \int_{D_0} \frac{(x_2 + x'_2)}{|x - \overline{x'}|^2} \omega(x) \omega(x') dx dx'$$

$$(\overline{x'} = (x'_1, -x'_2)) .$$

To show this we observe that since $\omega = \lambda I_{\{\psi > 0\}}$, integration by parts yields

$$0 = \int_{D_0} \omega\psi_{x_2} dx = \int_{D_0} \omega(G_0\omega)_{x_2} dx - Q \int_{D_0} \omega dx \; ;$$

thus, noting an obvious cancellation,

$$Q \int_{D_0} \omega dx = \int_{D_0} \int_{D_0} \frac{\partial}{\partial x_2} g_0(x,x')\omega(x)\omega(x')dxdx'$$

$$= \int_{D_0} \int_{D_0} \frac{\partial}{\partial x_2} h_0(x,x')\omega(x)\omega(x')dxdx' \; ,$$

which is (6.11).

The asymptotic analysis as $\lambda \to \infty$ remains valid for the solution in the alternate approach. Now we find the convergence (in the distributional sense)

$$\omega_{P,\lambda}(x) \to \delta(x - X_P^*) \quad \text{as} \quad \lambda \to \infty$$

where $X_P^* = (0,P)$ by virtue of the vortex impulse constraint (and (6.1)). Also, as a consequence of (6.11), we have

$$Q_{P,\lambda} \to Q_P^* = 1/4\pi P \quad \text{as} \quad \lambda \to \infty \; .$$

Finally, the limiting form of solutions as discussed in §4 remains equally valid for $\omega_{P,\lambda}$ as $\lambda \to \infty$.

APPENDIX: Vortex energy and impulse

We include this discussion in order to clarify the physical basis of the concepts of energy and impulse of an ideal fluid flow defined in §1. These concepts are developed in the literature only for the motion of a body through an irrotational flow or for a vortex motion in an unbounded fluid (without interior boundaries). The corresponding results in the more general context needed here - the motion of a body through a rotational flow - do not seem to be available in the standard references; see, for example, Batchelor [1].

We consider a body $\tilde{D}$ (compact, connected, simply-connected, smooth boundary) moving with translational velocity $U(t)$ and zero angular velocity through an ideal fluid at rest at infinity; the fluid occupies the exterior domain $D = R^2 \backslash \tilde{D}$. The (fluid) velocity field, $u = u(x,t)$, relative to a frame fixed in $\tilde{D}$ is governed by the equations

(A.1) $\quad \nabla \cdot u = 0 \;\text{ in }\; D, \quad \nu \cdot u = 0 \;\text{ on }\; \partial D,$

(A.2) $\quad u_t + u \cdot \nabla u = -\nabla p - \dot{U} \;\text{ in }\; D;$

ν denotes the unit normal on ∂D directed exterior to D. The (fluid) velocity field relative to a corresponding frame fixed at infinity is then given by

$$\hat{u} = \hat{u}(x,t) = u(x,t) + U(t) .$$

We shall require that

(A.3) $\quad |\hat{u}| = O(|x|^{-2}), \; |\nabla \hat{u}| = O(|x|^{-3}) \;\text{ as }\; |x| \to \infty;$

this restriction - adequate for our purposes - is more than is necessary for a general discussion of these concepts. The vorticity, $\omega = \omega(x,t)$, is given by

$$\omega = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} = \frac{\partial \hat{u}_2}{\partial x_1} - \frac{\partial \hat{u}_1}{\partial x_2} .$$

The dynamical equation (A.2) can be written in either of the following, well-known, equivalent forms:

(A.4) $$\hat{u}_t = \omega J u - \nabla b, \quad b = \frac{1}{2}|u|^2 + p ,$$

(A.5) $$\omega_t + u \cdot \nabla \omega = 0 ;$$

recall $J(a_1, a_2) = (a_2, -a_1)$.

We now define two functionals of any velocity field $\hat{u}$ satisfying (A.3) and the condition that ω has bounded support in D. The fluid energy is defined to be

(A.6) $$E^* = \frac{1}{2} \int_D |\hat{u}|^2 dx .$$

The fluid impulse is defined to be

(A.7) $$P^* = (P_1^*, P_2^*) \quad \text{with}$$

$$\begin{cases} P_1^* = \int_D x_2 \omega dx + \int_{\partial D} x_2 \hat{u} \cdot J\nu ds \\ \\ P_2^* = - \int_D x_1 \omega dx - \int_{\partial D} x_1 \hat{u} \cdot J\nu ds . \end{cases}$$

The physical significance of E^* is entirely standard; that of P^* is not however. In an appropriate sense P^* takes the place of the fluid momentum which cannot be defined directly because the integral

$$\int_D \hat{u} dx$$

is divergent according to (A.3). Indeed, if the lack of convergence is simply ignored then the expressions (A.7) can be derived formally from the latter integral by integration by parts.

In order to clarify the meaning of (A.7) we provide the following proposition in which the impulse is related to the reaction force of the fluid on the body $\tilde{D}$ when accelerated.

Proposition. Let $\hat{u}$ satisfy (A.1, A.2, A.3) and the condition that ω has bounded support in D. Then

$$F = -\int_{\partial D} p\nu ds = \frac{dP^*}{dt} . \tag{A.8}$$

We shall derive the first component of (A.8) only, as the other is similar; namely, we shall show

$$-\int_{\partial D} p\nu_1 ds = \frac{d}{dt}\left\{\int_D x_2 \omega dx + \int_{\partial D} x_2 \hat{u}\cdot J\nu ds\right\} . \tag{A.9}$$

Using the definition in (A.4) we can write

$$-\int_{\partial D} p\nu_1 ds = \int_{\partial D} \frac{1}{2}|u|^2 \nu_1 ds - \int_{\partial D} b\nu_1 ds ,$$

and so we consider these two integrals separately.

First integral:

$$\int_{\partial D} \frac{1}{2}|u|^2\nu_1 ds = \int_{\partial D} \frac{1}{2}|\hat{u}|^2 \nu_1 ds - \int_{\partial D} \hat{u}\cdot U\nu_1 ds$$

$$= \int_D \frac{\partial}{\partial x_1}\left(\frac{1}{2}|\hat{u}|^2\right)dx - \int_{\partial D} \hat{u}\cdot U\nu_1 ds .$$

Applying the identity $\frac{\partial}{\partial x_1}(\frac{1}{2}|\hat{u}|^2) = \omega\hat{u}_2 + \nabla\cdot(\hat{u}_1\hat{u})$ we then have

$$\int_{\partial D} \frac{1}{2}|u|^2\nu_1 ds = \int_D \omega\hat{u}_2 dx + \int_{\partial D} [(\nu\cdot U)\hat{u}_1 - (\hat{u}\cdot U)\nu_1]ds$$

$$= \int_D \omega u_2 dx + U_2\{\int_D \omega dx + \int_{\partial D} \hat{u}\cdot J\nu ds\}$$

$$= \int_D \omega u_2 dx \ ;$$

here we use the fact that the (total) circulation

$$\Gamma^* = \int_D \omega dx + \int_{\partial D} \hat{u}\cdot J\nu ds = 0 \tag{A.10}$$

since integration by parts is justified according to (A.3). Continuing we calculate

$$\int_D \omega u_2 dx = \int_D \nabla x_2\cdot(\omega u)dx$$

$$= -\int_D x_2 u\cdot\nabla\omega dx$$

$$= \int_D x_2\omega_t dx \ ,$$

by virtue of equation (A.5). Therefore,

$$\int_{\partial D} \frac{1}{2}|u|^2\nu_1 ds = \frac{d}{dt}\int_D x_2\omega dx \ . \tag{A.11}$$

Second integral:

$$- \int_{\partial D} b\nu_1 ds = \int_{\partial D} b \frac{\partial x_2}{\partial \tau} ds$$

$$= - \int_{\partial D} x_2 \frac{\partial b}{\partial \tau} ds$$

$$= \int_{\partial D} x_2(\hat{u}_t - \omega J u)\cdot \tau ds$$

$$= \int_{\partial D} x_2 \hat{u}_t \cdot \tau ds$$

with $\tau = J\nu$ (unit tangent) on ∂D; here we use integration by parts around ∂D followed by equation (A.4). Therefore,

$$\text{(A.12)} \qquad - \int_{\partial D} b\nu_1 ds = \frac{d}{dt} \int_{\partial D} x_2 \hat{u}\cdot J\nu ds \; .$$

The required identity (A.9) now follows by combining (A.11) and (A.12).

The term "impulse" is derived from another interpretation of P^*. We say that $\hat{u}$ is expressed in terms of impulsive pressures, p^*, and impulsive (body) forces, f^*, if it can be written

$$\text{(A.13)} \qquad \hat{u} = -\nabla p^* + f^* \quad \text{with} \quad |f^*| = O(|x|^{-2-\varepsilon})$$
$$\text{as} \quad |x| \to \infty, \; \varepsilon > 0 \; .$$

It then follows that

$$\text{(A.14)} \qquad P^* = - \int_{\partial D} p^* \nu ds + \int_D f^* dx \; ,$$

and this expression is independent of the choice taken in (A.13). A further discussion is given in Lamb [3] §1.11.

In the case that the body is assumed to be symmetric about the x_1 axis and the flow likewise the expressions (A.6) and (A.7) take an especially simple form; this is the situation considered in §5. In keeping with the notation established earlier (see §5) we shall now let $\tilde{D}$ and D denote respectively the intersection of the body and the (fluid) domain with the upper half plane $\{x_2 > 0\}$. We assume also that $U = Q(1,0)$, $0 < Q < \infty$. These symmetry assumptions (along with (A.3)) imply that $\omega(\bar{x}) = -\omega(x)$, $\hat{u}_1(\bar{x}) = \hat{u}_1(x)$, $\hat{u}_2(\bar{x}) = -\hat{u}_2(x)$ for $\bar{x} = (x_1,-x_2)$, $x \in D$. Thus, we find

(A.6') $$\frac{1}{2} E^* = \frac{1}{2} \int_D |\hat{u}|^2 dx ,$$

(A.7') $$\frac{1}{2} P_1^* = \int_D x_2 \omega dx + \int_{\partial D} x_2 \hat{u} \cdot J\nu ds, \quad P_2^* = 0 .$$

We claim that these quantities can be expressed in the alternate forms:

(A.15) $$\frac{1}{2} E^* = \frac{1}{2} \int_D \omega G \omega dx + \frac{1}{2} MQ^2 ,$$

(A.16) $$\frac{1}{2} P_1^* = \int_D \eta \omega dx + MQ, \quad P_2^* = 0 ,$$

where the induced mass of $\tilde{D}$ is defined by

(A.17) $$M = \int_D |\nabla(\eta - x_2)|^2 dx \ ;$$

we retain the definitions of G and η from §5.

In the derivations of these identities we let $\hat{\eta} = \eta - x_2$; then the streamfunction for the velocity field $\hat{u}$ is given by $\hat{\psi} = G\omega - Q\hat{\eta}$.

Derivation of (A.15):

$$\frac{1}{2} E^* = \frac{1}{2} \int_D |\nabla\hat{\psi}|^2 dx$$

$$= \frac{1}{2} \int_D |\nabla G\omega|^2 dx - Q \int_D \nabla G\omega \cdot \nabla\hat{\eta} dx + \frac{1}{2} Q^2 \int_D |\nabla\hat{\eta}|^2 dx$$

$$= \frac{1}{2} \int_D \omega G\omega dx + \frac{1}{2} Q^2 \int_D |\nabla\hat{\eta}|^2 dx \ .$$

Derivation of (A.16):

$$\frac{1}{2} P_1^* = \int_D \eta\omega dx + \int_D \hat{\eta}\Delta\hat{\psi} dx - \int_{\partial D} \hat{\eta} \frac{\partial\hat{\psi}}{\partial\nu} ds$$

$$= \int_D \eta\omega dx - \int_{\partial D} \hat{\psi} \frac{\partial\hat{\eta}}{\partial\nu} ds$$

$$= \int_D \eta\omega dx + Q \int_{\partial D} \hat{\eta} \frac{\partial\hat{\eta}}{\partial\nu} ds$$

$$= \int_D \eta\omega dx + Q \int_D |\nabla\hat{\eta}|^2 dx \ .$$

As a consequence of these identities we see that the energy and impulse may both be split into a "rotational part" and an "irrotational part"; the functionals $E(\omega)$ and $P(\omega)$ defined by (1.10) and

(1.11) represent the rotational parts respectively, while $\frac{1}{2} MQ^2$ and MQ with definition (A.17) represent the irrotational parts respectively.

REFERENCES

1. Batchelor, G. K., An introduction to fluid dynamics. Cambridge University Press, 1974.

2. Friedman, A. and Turkington, B., Vortex rings: existence and asymptotic estimates. Trans. Amer. Math. Soc. 268 (1981), 1-37.

3. Lamb, H., Hydrodynamics. Cambridge University Press, 1932.

4. Turkington, B. On steady vortex flow in two dimensions, I.

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We extend the results of our previous work (Part I) concerning steady ideal flow with vorticity in a bounded domain to the situation where the domain is unbounded and the flow is uniform at infinity. Prototypical examples include vortex pairs and Föppl vortex wakes behind a cylinder (in uniform translation). The existence of solutions and the asymptotic behavior of these solutions in a

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certain singular limit are established as in Part I using a direct variational method. The variational principle needed here is rather nonstandard, and so a detailed discussion of its formulation is given. Special difficulties arise for flows in an unbounded domain due to a lack of compactness (in the appropriate function space); consequently, we find that there is nonexistence of solutions in some cases.